

A GOLOD COMPLEX WITH NON-SUSPENSION MOMENT-ANGLE COMPLEX

KOUYEMON IRIYE AND TATSUYA YANO

ABSTRACT. It could be expected that the moment-angle complex associated with a Golod simplicial complex is homotopy equivalent to a suspension space. In this paper, we provide a counter example to this expectation. We have discovered this complex through the studies of the Golod property of the Alexander dual of a join of simplicial complexes, and that of a union of simplicial complexes.

1. INTRODUCTION

The *Stanley-Reisner ring* (or *face ring*) of a simplicial complex K over an index set $[m] = \{1, \dots, m\}$ is defined as the quotient graded algebra

$$\mathbf{k}[K] = \mathbf{k}[v_1, \dots, v_m] / \mathcal{I}_K,$$

where \mathbf{k} is a commutative ring with unit and $\mathcal{I}_K = (v_{i_1} \cdots v_{i_k} \mid \{i_1, \dots, i_k\} \notin K)$ is the *Stanley-Reisner ideal* of K . K is called *Golod* over a field \mathbf{k} if its Stanley-Reisner ring $\mathbf{k}[K]$ is Golod over \mathbf{k} . That is, the multiplication and all higher Massey products in $\mathrm{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) = H(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[K], d)$ are trivial, where the Koszul differential algebra $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[K], d)$ is the bigraded differential algebra with $\deg u_i = (1, 2)$, $\deg v_i = (0, 2)$, and $du_i = v_i$ for $i = 1, \dots, m$. Originally, the algebra $\mathbf{k}[K]$ or the ideal \mathcal{I}_K was defined to be Golod if the following equation holds:

$$\sum_{i \geq 0; j \geq 0} \dim_{\mathbf{k}} \mathrm{Tor}_{j, 2i}^{\mathbf{k}[K]}(\mathbf{k}, \mathbf{k}) t^j z^i = \frac{(1 + tz)^n}{1 - t \sum_{i \geq 0; j \geq 1} \dim_{\mathbf{k}} \mathrm{Tor}_{j, 2i}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) t^j z^i},$$

where $\mathrm{Tor}_{j, 2i}^{\mathbf{k}[K]}(\mathbf{k}, \mathbf{k})$ and $\mathrm{Tor}_{j, 2i}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$ denote the homogeneous components of degree $2i$. Golod [8] proved the equivalence of the two conditions, and thereafter his name has been used to refer a ring that satisfies the condition. The reader may also refer to Gulliksen and Levin [10] or Avramov [1].

Baskakov, Buchstaber, and Panov [3] and Franz [7] independently demonstrated that the torsion algebra $\mathrm{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$ is isomorphic to the cohomology ring of the moment-angle complex Z_K associated with K .

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Theorem 1.1([3, 7]). *For a commutative ring \mathbf{k} with unit, the following isomorphisms of algebras hold:*

$$H^*(Z_K; \mathbf{k}) \cong \mathrm{Tor}_*^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) \cong \bigoplus_{I \subset [m]} \tilde{H}^*(K_I; \mathbf{k}),$$

where $\tilde{H}^*(K_I; \mathbf{k})$ denotes the reduced cohomology of the full subcomplex K_I of K on I , and $\tilde{H}^*(K_\emptyset; \mathbf{k}) = 0$ for $*$ $\neq -1$ and $= \mathbf{k}$ for $*$ $= -1$. The last isomorphism is the sum of isomorphisms given by

$$H^p(Z_K; \mathbf{k}) \cong \bigoplus_{I \subset [m]} \tilde{H}^{p-|I|-1}(K_I; \mathbf{k}),$$

and the ring structure is given by the maps

$$\tilde{H}^{p-|I|-1}(K_I; \mathbf{k}) \otimes \tilde{H}^{q-|J|-1}(K_J; \mathbf{k}) \rightarrow \tilde{H}^{p+q-|I|-|J|-1}(K_{I \cup J}; \mathbf{k})$$

that are induced by the canonical simplicial maps $K_{I \cup J} \rightarrow K_I * K_J$ for $I \cap J = \emptyset$ and zero otherwise, where $K_I * K_J$ denotes the join of two simplicial complexes K_I and K_J .

Here, we recall that if the moment-angle complex Z_K is (homotopy equivalent to) a suspension, then the multiplication and all higher Massey products in $H^*(Z_K; \mathbf{k})$ are trivial. For example, see Corollary 3.11 of [22]. That is, the following implication holds:

$$(1.1) \quad Z_K \text{ is a suspension} \implies K \text{ is Golod},$$

where K is Golod if K is Golod over any field \mathbf{k} . This observation enables us to investigate the Golod property through the study of moment-angle complexes. One of the first studies in this direction was introduced by Grbić and Theriault [9]. They demonstrated that the moment-angle complex associated with a shifted simplicial complex is homotopy equivalent to a wedge of spheres. In [14], Kishimoto and the first author extended this result to dual sequentially Cohen-Macaulay complexes, and provided some new Golod complexes. In these studies, the following theorem concerning the decomposition of polyhedral products (see Definition 2.1), as introduced by Bahri, Bendersky, Cohen, and Gitler [2], plays an essential role.

Theorem 1.2([2]). *Let K be a simplicial complex on an index set $[m]$ and let $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i \in [m]}$, where each X_i is a based space and CX is the reduced cone of a based space X . Then, the following homotopy equivalence holds:*

$$\Sigma Z_K(C\underline{X}, \underline{X}) \simeq \Sigma \bigvee_{I \subset [m]} \Sigma |K_I| \wedge \hat{X}^I,$$

where $\hat{X}^I = \bigwedge_{i \in I} X_i$ and $\hat{X}^\emptyset = *$.

We call this decomposition of polyhedral products the BBCG decomposition for K . If this decomposition is desuspendable, i.e., if the homotopy equivalence

$$Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{I \subset [m]} \Sigma |K_I| \wedge \hat{X}^I$$

holds for any sequence of based CW-complexes \underline{X} , then we say that the BBCG decomposition is *desuspendable* for K .

In this paper, we study the Golod properties of the Alexander dual of $K * L$ and $K \cup_\alpha L$, where α is a common face of K and L . The precise statements of the results are given in the next section.

By Theorem 1.1, the multiplicative structure of $H^*(Z_K; \mathbf{k})$ is trivial if and only if the maps $K_{I \cup J} \rightarrow K_I * K_J$ for $I \cap J = \emptyset$ induce the trivial maps on the reduced cohomology theory. By strengthening this condition, K is said to be *(stably) homotopy Golod* [14] if the maps $|K_{I \cup J}| \rightarrow |K_I * K_J|$ for $I \cap J = \emptyset$ are (stably) null homotopic and $H^*(Z_K; \mathbf{k})$ has trivial higher Massey products for any fields \mathbf{k} . By definition, the following implication holds:

$$(\text{stably}) \text{ homotopy Golod} \implies \text{Golod}.$$

The second purpose of this paper is to prove that this implication is strict.

Theorem 1.3. *There is a Golod simplicial complex K such that K is not stably homotopy Golod. Moreover, Z_K can be chosen to be torsion free.*

Here, a space or a simplicial complex X is called *torsion free* if its integral homology groups $H_*(X; \mathbb{Z})$ are torsion free.

It could be expected that the converse of the implication (1.1) is also true. Theorem 1.3 provides a counter example to this expectation. In fact, if Z_K is a suspension then the fat wedge filtration of Z_K is trivial, by Theorem 1.3 of [14]. By Theorem 6.9 of the same paper, we see that K is stably homotopy Golod, which contradicts our result. Thus, Z_K is not a suspension.

In the next section, we state the main results of this paper. The subsequent sections are devoted to their proofs.

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2. RESULTS

In this section, we state our main results. We begin by setting notation regarding simplicial complexes.

Let K be a simplicial complex on an index set V . In this paper, we only consider finite simplicial complexes. The subset $V(K)$ of V defined by $V(K) = \cup_{\sigma \in K} \sigma$ is called the vertex set of K , and an element of $V - V(K)$ is called a ghost vertex. For a finite set V , we denote the full simplex on V by Δ^V . Its boundary is denoted by $\partial \Delta^V$. We also use the symbol Δ^n to denote an n -dimensional simplex. $|K|$ denotes a geometric realization of K . The link and

star of a face σ of K are denoted by $\text{link}_K(\sigma)$ and $\text{star}_K(\sigma)$, respectively. For a subset $I \subset V$, K_I denotes the full subcomplex of K indexed by I ; that is, $K_I = \{\sigma \subset I \mid \sigma \in K\}$. Then, K_I is called the restriction of K to I . For a vertex v of K , the deletion of v from K is denoted by $K - v = K_{V-\{v\}}$. For simplicial complexes K and L with disjoint index sets V and W , the simplicial *join* $K * L$ on the index set $V \sqcup W$ is defined by $K * L = \{\sigma \sqcup \tau \subset V \sqcup W \mid \sigma \in K, \tau \in L\}$, where \sqcup always denotes the disjoint union of sets. We write $K * c$ to stand for $K * \Delta^{\{c\}}$. The (simplicial) *Alexander dual* K_V^\vee is defined by $K_V^\vee = \{\sigma \subset V \mid \sigma^c = V - \sigma \notin K\}$. If $V = V(K)$ or V is clear from a context, we simply write K^\vee for K_V^\vee . It is easy to see that for a subset I of V , I is a facet of K if and only if I^c is a minimal-non face of K^\vee . The restriction of K^\vee and the link of K are related by the following formula: $(K^\vee)_I = (\text{link}_K(I^c))^\vee$.

Next we review polyhedral products, which are a generalization of a moment-angle complexes.

Definition 2.1. Let K be a simplicial complex on $[m]$, and $(\underline{X}, \underline{A})$ be a sequence of pairs of based spaces $\{(X_i, A_i)\}_{i \in [m]}$. The polyhedral product $Z_K(\underline{X}, \underline{A})$ is defined by

$$Z_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \quad (\subset X_1 \times \cdots \times X_m),$$

where $(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m$, with $Y_i = X_i$ for $i \in \sigma$ and A_i for $i \notin \sigma$. If $(X_i, A_i) = (X, A)$ for all $i \in [m]$, then we write $Z_K(X, A)$ for $Z_K(\underline{X}, \underline{A})$.

Example 2.2. In [21], Porter used a polyhedral product to define a higher order Whitehead product, and proved the following homotopy equivalence:

$$\begin{aligned} Z_{\partial \Delta[m]}(C\underline{X}, \underline{X}) &= \bigcup_{i=1}^m CX_1 \times \cdots \times CX_{i-1} \times X_i \times CX_{i+1} \times \cdots \times CX_m \\ &\simeq X_1 * \cdots * X_m \\ &\simeq \Sigma^{m-1} X_1 \wedge \cdots \wedge X_m. \end{aligned}$$

The polyhedral product $Z_K(D^2, S^1)$ is the moment-angle complex of K , and is written simply as Z_K . We refer the reader to [14, 5] for further examples of polyhedral products. In this paper, we are interested in the homotopy types of the polyhedral products $Z_K(C\underline{X}, \underline{X})$.

Theorem 2.3. *Let K and L be simplicial complexes on disjoint index sets V and W , respectively. If $K \neq \Delta^V$ and $L \neq \Delta^W$, then the BBCG decomposition is desuspendable for $(K * L)^\vee$.*

Corollary 2.4. *Let K and L be simplicial complexes on disjoint index sets V and W , respectively. If $K \neq \Delta^V$ and $L \neq \Delta^W$, then $(K * L)^\vee$ is Golod.*

Because the Stanley-Reisner ideal of $(K * L)^\vee$ is the product of those of K^\vee and L^\vee , this corollary provides a topological proof of a classical result given in [12]. The reader may also

refer to [6] and [16]. We also remark that $(\Delta^V * L)^\vee$ is Golod if and only if L^\vee is Golod, because $(\Delta^V * L)^\vee = \Delta^V * L^\vee$ (see Lemma 3.1).

Theorem 2.5. *Let K and L be simplicial complexes of non-negative dimension on index sets V and W , respectively. Assume that $\alpha = V \cap W$ is a common face of K and L . If $K \neq \Delta^V$ or $L \neq \Delta^W$, and α is not a facet of K or L , then the BBCG decomposition is desuspendable for $(K \cup_\alpha L)^\vee$.*

Corollary 2.6. *Let K and L be simplicial complexes that satisfy the same conditions as stated in Theorem 2.5. If α is not a facet of K or L , then $(K \cup_\alpha L)^\vee$ is Golod.*

It is natural to ask whether $(K \cup_\alpha L)^\vee$ is still Golod even when α is a facet of K or L in Corollary 2.6. In general, the answer is that this does not hold. Whether it does depend on K , L , and α . In fact, we can construct many simplicial complexes $(\Delta^V \cup_\alpha L)^\vee$ that are not Golod, such as in Example 5.3.4. Incidentally, $(\Delta^V \cup_\alpha \Delta^W)^\vee$ is a non-Golod simplicial complex, where $(\Delta^V \cup_\alpha \Delta^W)^\vee = \partial \Delta^{V-\alpha} * \Delta^\alpha * \partial \Delta^{W-\alpha}$ (see Lemma 4.1).

To construct a simplicial complex satisfying Theorem 1.3, we first need to fix some notation.

Let K be a simplicial complex on an index set V , with facets F_1, \dots, F_k . We take new vertices v_1, \dots, v_k , and define a new simplicial complex $F(K)$ on the index set $V \sqcup \{v_1, \dots, v_k\}$ with facets $F_1 + v_1, \dots, F_k + v_k$, where $F_i + v_i = F_i \sqcup \{v_i\}$ as usual. Then, K is a subcomplex of $F(K)$ and $|K|$ is a deformation retract of $|F(K)|$.

For two simplicial complexes K and L , we define a “product” $K \boxtimes L$ as follows. Define a linear order \leq on the vertex sets of K and L . The vertex set of $K \boxtimes L$ is $V(K) \times V(L)$. An n -simplex is a set $\{(x_0, y_0), \dots, (x_n, y_n)\}$ such that $x_0 \leq \dots \leq x_n$, $y_0 \leq \dots \leq y_n$, $\{x_0, \dots, x_n\}$ is a simplex of K , and $\{y_0, \dots, y_n\}$ is a simplex of L . It is well-known that $|K \boxtimes L|$ is homeomorphic to $|K| \times |L|$. If v is a vertex of L , then the subcomplex $K \boxtimes \Delta^{\{v\}}$ of $K \boxtimes L$ is abbreviated as $K \boxtimes v$.

It follows from the simplicial approximation theorem that there is a simplicial map $\eta_k : S_k^3 \rightarrow S_4^2$ whose geometrical realization $|\eta_k| : |S_k^3| \rightarrow |S_4^2|$ is homotopic to the Hopf map $\eta : S^3 \rightarrow S^2$, where S_k^n denotes a triangulation of an n -sphere S^n with k -vertices. In fact, we can choose $k = 12$ in this case, by [18].

We consider the simplicial set Δ^1 as the full simplex on $[2]$. By $S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2$ we denote the simplicial complex obtained from the disjoint union of two simplicial complexes $S_k^3 \boxtimes \Delta^1 \sqcup S_4^2$, which is defined by identifying $(v, 2) \in V(S_k^3) \times [2]$ with $\eta_k(v) \in V(S_4^2)$. We embed S_k^3 into $S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2$ by applying the map $v \mapsto (v, 1)$.

We set α to be the vertex set of $(S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2) \cup F(S_k^3)$. Finally, we take new vertices v_0, w_1, w_2 and set

$$K = \Delta^{\alpha+v_0} \cup ((S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2) \cup F(S_k^3)) * \Delta^{\{w_1\}} \cup S_k^3 * \Delta^{\{w_1, w_2\}},$$

which is the union of $\Delta^{\alpha+v_0}$ and $\Delta^\alpha \cup ((S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2) \cup F(S_k^3)) * \Delta^{\{w_1\}} \cup S_k^3 * \Delta^{\{w_1, w_2\}}$.

Theorem 2.7. K^\vee is a Golod simplicial complex that is not stably homotopy Golod. Moreover, Z_{K^\vee} is torsion free if K is constructed from the map $\eta_{12} : S_{12}^3 \rightarrow S_4^2$ defined in [18].

3. PROOF OF THEOREM 2.3.

In this section, we prove Theorem 2.3. We begin by stating some elementary lemmas, for which the proofs are omitted.

Lemma 3.1. Let K and L be simplicial complexes with disjoint index sets V and W , respectively. Then, $(K * L)_{V \sqcup W}^\vee = K_V^\vee * \Delta^W \cup \Delta^V * L_W^\vee$.

Lemma 3.2. Let K_i and L_i for $i = 1, 2$ be simplicial complexes with disjoint index sets. Then, $(K_1 * L_1) \cap (K_2 * L_2) = (K_1 \cap K_2) * (L_1 \cap L_2)$.

Lemma 3.3. Let K be a simplicial complex with two subcomplexes K_1 and K_2 . If $K = K_1 \cup K_2$, then $Z_K(C\underline{X}, \underline{X}) = Z_{K_1}(C\underline{X}, \underline{X}) \cup Z_{K_2}(C\underline{X}, \underline{X})$ and $Z_{K_1}(C\underline{X}, \underline{X}) \cap Z_{K_2}(C\underline{X}, \underline{X}) = Z_{K_1 \cap K_2}(C\underline{X}, \underline{X})$.

Lemma 3.4. Let K and L be simplicial complexes with disjoint index sets V and W , respectively. Then,

$$Z_{K * L}(C\underline{X}, \underline{X}) = Z_K(C\underline{X}_V, \underline{X}_V) \times Z_L(C\underline{X}_W, \underline{X}_W),$$

where \underline{X}_V and \underline{X}_W are the sub-sequences of \underline{X} indexed by V and W , respectively.

Proposition 3.5. Let K be a simplicial complex on $[m]$ and \underline{X} be a sequence of based CW-complexes. If $Z_K(C\underline{X}, \underline{X})$ is a simply connected co- H -space, then $Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{I \subset [m]} \Sigma |K_I| \wedge \widehat{X}^I$.

Proof. First, we remark that $\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \widehat{X}^I$ is also simply connected. By Theorem 1.2 we have that $H_1(\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \widehat{X}^I) \cong H_1(Z_K(C\underline{X}, \underline{X})) = 0$. Because $\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \widehat{X}^I$ is a suspension, its fundamental group is a free group. Thus, $\pi_1(\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \widehat{X}^I) = 0$.

For a subset $I \subset [m]$, the canonical projection $p_I : \prod_{i \in [m]} CX_i \rightarrow \prod_{i \in I} CX_i$ induces a map $Z_K(C\underline{X}, \underline{X}) \rightarrow Z_{K_I}(C\underline{X}_I, \underline{X}_I)$, which is also denoted by p_I . We define a subset $Z_{K_I}(C\underline{X}_I, \underline{X}_I)'$ of $Z_{K_I}(C\underline{X}_I, \underline{X}_I)$ by the equation

$$Z_{K_I}(C\underline{X}_I, \underline{X}_I)' = \{(x_{i_1}, \dots, x_{i_k}) \in Z_{K_I}(C\underline{X}_I, \underline{X}_I) \mid x_{i_j} = * \text{ for some } j\},$$

where $I = \{i_1, \dots, i_k\}$. In [14], it is shown that $Z_{K_I}(C\underline{X}_I, \underline{X}_I)/Z_{K_I}(C\underline{X}_I, \underline{X}_I)' \simeq \Sigma|K_I| \wedge \widehat{X}^I$. Now, we consider the composite of maps

$$\begin{aligned} f : Z_K(C\underline{X}, \underline{X}) &\rightarrow \bigvee_{I \subset [m]}^{2^m} Z_K(C\underline{X}, \underline{X}) \xrightarrow{\bigvee_{I \subset [m]} P_I} \bigvee_{I \subset [m]} Z_{K_I}(C\underline{X}_I, \underline{X}_I) \\ &\rightarrow \bigvee_{I \subset [m]} Z_{K_I}(C\underline{X}_I, \underline{X}_I)/Z_{K_I}(C\underline{X}_I, \underline{X}_I)' \simeq \bigvee_{I \subset [m]} \Sigma|K_I| \wedge \widehat{X}^I, \end{aligned}$$

where the first map is the iterated co-multiplication of $Z_K(C\underline{X}, \underline{X})$. Because $Z_K(C\underline{X}, \underline{X})$ and $\bigvee_{I \subset [m]} \Sigma|K_I| \wedge \widehat{X}^I$ are simply connected CW-complexes, to prove that f is homotopy equivalent it suffices to show that f induces a homology isomorphism. In [15], it is shown that Σf is a homotopy equivalence. In particular, f induces a homology isomorphism, and thus we complete the proof. \square

Proof of Theorem 2.3. By Lemmas 3.1 and 3.2, we have that $(K * L)^\vee = K^\vee * \Delta^W \cup \Delta^V * L^\vee$ and $K^\vee * \Delta^{V(L)} \cap \Delta^{V(K)} * L^\vee = K^\vee * L^\vee$. Therefore, from Lemma 3.3 we obtain the following push-out diagram of spaces:

$$\begin{array}{ccc} Z_{K^\vee * L^\vee}(C\underline{X}, \underline{X}) & \longrightarrow & Z_{K^\vee * \Delta^W}(C\underline{X}, \underline{X}) \\ \downarrow & & \downarrow \\ Z_{\Delta^V * L^\vee}(C\underline{X}, \underline{X}) & \longrightarrow & Z_{(K * L)^\vee}(C\underline{X}, \underline{X}). \end{array}$$

Here, we remark that $K^\vee * L^\vee$ and $K^\vee * \Delta^W$ are non-void simplicial complexes, because we assume that $K \neq \Delta^V$ and $L \neq \Delta^W$. By Lemma 3.4, the above push-out diagram is equivalent to the following push-out diagram:

$$\begin{array}{ccc} Z_{K^\vee}(C\underline{X}_V, \underline{X}_V) \times Z_{L^\vee}(C\underline{X}_W, \underline{X}_W) & \longrightarrow & Z_{K^\vee}(C\underline{X}_V, \underline{X}_V) \times \prod_{w \in W} CX_w \\ \downarrow & & \downarrow \\ \prod_{v \in V} CX_v \times Z_{L^\vee}(C\underline{X}_W, \underline{X}_W) & \longrightarrow & Z_{(K * L)^\vee}(C\underline{X}, \underline{X}). \end{array}$$

Because $\prod_{v \in V} CX_v$ and $\prod_{w \in W} CX_w$ are contractible, the above diagram yields the following homotopy equivalences:

$$\begin{aligned} Z_{(K * L)^\vee}(C\underline{X}, \underline{X}) &\simeq Z_{K^\vee}(C\underline{X}_V, \underline{X}_V) * Z_{L^\vee}(C\underline{X}_W, \underline{X}_W) \\ &\simeq \Sigma Z_{K^\vee}(C\underline{X}_V, \underline{X}_V) \wedge Z_{L^\vee}(C\underline{X}_W, \underline{X}_W). \end{aligned}$$

By Theorem 1.2, $\Sigma Z_{K^\vee}(C\underline{X}_V, \underline{X}_V)$ is a double suspension, which implies that $Z_{(K * L)^\vee}(C\underline{X}, \underline{X})$ is also a double suspension. By invoking Proposition 3.5, we complete the proof. \square

4. PROOF OF THEOREM 2.5.

In this section, we prove Theorem 2.5. Again, we begin by stating some elementary lemmas.

Lemma 4.1. *Let K and L be simplicial complexes without ghost vertices on index sets V and W , respectively. Let $\alpha = V \cap W$ be a common face of K and L . Then, $(K \cup_\alpha L)^\vee = (\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (K^\vee * \Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee)$.*

Proof. For $u \in V - \alpha$ and $v \in W - \alpha$, $\{u, v\}$ is a minimal non-face of $K \cup_\alpha L$, and a minimal non-face of K or L is also a minimal non-face of $K \cup_\alpha L$. For any $u \in V - \alpha$ and $v \in V - \alpha$, $V \cup W - \{u, v\}$ is a facet of $(K \cup_\alpha L)^\vee$. Furthermore, for any minimal non-face σ of K or L , $V \cup W - \sigma$ is a facet of $(K \cup_\alpha L)^\vee$. This implies the desired equality of the two complexes. \square

Lemma 4.2. *If K is a simplicial complex with a ghost vertex, then the BBCG decomposition for K^\vee is desuspendable. In particular, K^\vee is Golod.*

Proof. Let K be a simplicial complex on $[m]$ and v be a ghost vertex of K . Then, $[m] - v$ is a facet of K^\vee , and thus $\dim K^\vee \geq m - 2$. Then, it follows from Theorem 1.2 and Proposition 3.5 of [14] that the BBCG decomposition for K^\vee is desuspendable. \square

Lemma 4.3. *Let α be a face of a simplicial complex K on an index set V . If α is not a facet of K , then the inclusion map*

$$Z_{K^\vee}(C\underline{X}, \underline{X}) \rightarrow Z_{(\Delta^\alpha)^\vee}(C\underline{X}, \underline{X}) = Z_{\partial\Delta^{V-\alpha} * \Delta^\alpha}(C\underline{X}, \underline{X})$$

is null homotopic.

Proof. Because α is not a facet of K , there is a face β of K such that $\alpha \subsetneq \beta$. Then, $\Delta^\alpha \subsetneq \Delta^\beta \subset K$, which implies that $(\Delta^\alpha)^\vee \supsetneq (\Delta^\beta)^\vee \supset K^\vee$. That is, $\partial\Delta^{V-\alpha} * \Delta^\alpha \supsetneq \partial\Delta^{V-\beta} * \Delta^\beta \supset K^\vee$. Therefore, the inclusion $Z_{K^\vee}(C\underline{X}, \underline{X}) \hookrightarrow Z_{\partial\Delta^{V-\alpha} * \Delta^\alpha}(C\underline{X}, \underline{X})$ factors as $Z_{K^\vee}(C\underline{X}, \underline{X}) \rightarrow Z_{\partial\Delta^{V-\beta} * \Delta^\beta}(C\underline{X}, \underline{X}) \rightarrow Z_{\partial\Delta^{V-\alpha} * \Delta^\alpha}(C\underline{X}, \underline{X})$. To show that the inclusion $Z_{K^\vee}(C\underline{X}, \underline{X}) \rightarrow Z_{\partial\Delta^{V-\alpha} * \Delta^\alpha}(C\underline{X}, \underline{X})$ is null homotopic, it is sufficient to show that $Z_{\partial\Delta^{V-\beta} * \Delta^\beta}(C\underline{X}, \underline{X}) \rightarrow Z_{\partial\Delta^{V-\alpha} * \Delta^\alpha}(C\underline{X}, \underline{X})$ is null homotopic. Because

$$\begin{aligned} Z_{\partial\Delta^{V-\beta} * \Delta^\beta}(C\underline{X}, \underline{X}) &= Z_{\partial\Delta^{V-\beta}}(C\underline{X}_{V-\beta}, \underline{X}_{V-\beta}) \times \prod_{j \in \beta} CX_j, \\ Z_{\partial\Delta^{V-\alpha} * \Delta^\alpha}(C\underline{X}, \underline{X}) &= Z_{\partial\Delta^{V-\alpha}}(C\underline{X}_{V-\alpha}, \underline{X}_{V-\alpha}) \times \prod_{j \in \alpha} CX_j, \end{aligned}$$

it suffices to show that the map $f : Z_{\partial\Delta^{V-\beta}}(C\underline{X}_{V-\beta}, \underline{X}_{V-\beta}) \rightarrow Z_{\partial\Delta^{V-\alpha}}(C\underline{X}_{V-\alpha}, \underline{X}_{V-\alpha})$ induced by the inclusion of simplicial sets $\partial\Delta^{V-\beta} \rightarrow \partial\Delta^{V-\alpha}$ is null-homotopic. Because $\Delta^{V-\beta} \subset \partial\Delta^{V-\alpha}$, the above map factors through a contractible space $Z_{\Delta^{V-\beta}}(C\underline{X}_{V-\beta}, \underline{X}_{V-\beta})$, which implies that f is null-homotopic, and thus we complete the proof. \square

In addition to the above, we require the following lemma to prove Theorem 2.5.

Lemma 4.4 (Lemma 3.2 of [13]). Define Q as the push-out

$$\begin{array}{ccc} A \times (B \vee C) & \xrightarrow{\iota \times (1 \vee 1)} & CA \times (B \vee C) \\ \downarrow 1 \times (1 \vee *) & & \downarrow \\ A \times (B \vee D) & \longrightarrow & Q, \end{array}$$

where $\iota : A \rightarrow CA$ is the inclusion. Then, the homotopy equivalence

$$Q \xrightarrow{\simeq} B \vee \Sigma(A \wedge C) \vee (A \ltimes D)$$

holds, which is natural with respect to A , B , C , and D , where $X \ltimes Y = (X \times Y)/(X \times *)$.

Proof of Theorem 2.5. If K or L has a ghost vertex, then $K \cup_\alpha L$ also has a ghost vertex. In this case, the BBCG decomposition is desuspendable, by Lemma 4.2. Therefore, we assume that K and L do not have any ghost vertices. In the following proof, $Z_K(C\underline{X}, \underline{X})$ is abbreviated as Z_K .

First, we will show that if $K = \Delta^V$ and $L \neq \Delta^W$, then

$$Z_{(\Delta^V \cup_\alpha L)^\vee} \simeq (Z_{\partial \Delta^{V-\alpha}} \ltimes Z_{\partial \Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial \Delta^{V-\alpha}} \wedge Z_{L^\vee}).$$

It follows from Lemma 4.1 that $(\Delta^V \cup_\alpha L)^\vee = (\partial \Delta^{V-\alpha} * \Delta^\alpha * \partial \Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee)$. Then, by Lemma 3.3 we have the push-out diagram of spaces

$$\begin{array}{ccc} Z_{\partial \Delta^{V-\alpha} * L^\vee} & \longrightarrow & Z_{\Delta^{V-\alpha} * L^\vee} \\ \downarrow & & \downarrow \\ Z_{\partial \Delta^{V-\alpha} * \Delta^\alpha * \partial \Delta^{W-\alpha}} & \longrightarrow & Z_{(\Delta^V \cup_\alpha L)^\vee}, \end{array}$$

which by Lemma 3.4 is equivalent to the following push-out diagram:

$$\begin{array}{ccc} Z_{\partial \Delta^{V-\alpha}} \times Z_{L^\vee} & \longrightarrow & Z_{\Delta^{V-\alpha}} \times Z_{L^\vee} \\ \text{id} \times \text{incl} \downarrow & & \downarrow \\ Z_{\partial \Delta^{V-\alpha}} \times Z_{\Delta^\alpha} \times Z_{\partial \Delta^{W-\alpha}} & \longrightarrow & Z_{(\Delta^V \cup_\alpha L)^\vee}. \end{array}$$

By Lemma 4.3, the inclusion map $Z_{L^\vee} \hookrightarrow Z_{\Delta^\alpha} \times Z_{\partial \Delta^{W-\alpha}}$ is null-homotopic. Therefore, $Z_{(\Delta^V \cup_\alpha L)^\vee}$ is homotopy equivalent to the push-out P of the following diagram:

$$(4.1) \quad \begin{array}{ccc} Z_{\partial \Delta^{V-\alpha}} \times Z_{L^\vee} & \longrightarrow & Z_{\Delta^{V-\alpha}} \times Z_{L^\vee} \\ \text{id} \times * \downarrow & & \downarrow \\ Z_{\partial \Delta^{V-\alpha}} \times Z_{\Delta^\alpha} \times Z_{\partial \Delta^{W-\alpha}} & \xrightarrow{j} & P \end{array}$$

By Lemma 4.4, P is homotopy equivalent to

$$(Z_{\partial \Delta^{V-\alpha}} \ltimes (Z_{\Delta^\alpha} \times Z_{\partial \Delta^{W-\alpha}})) \vee \Sigma(Z_{\partial \Delta^{V-\alpha}} \wedge Z_{L^\vee}) \simeq (Z_{\partial \Delta^{V-\alpha}} \ltimes Z_{\partial \Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial \Delta^{V-\alpha}} \wedge Z_{L^\vee}),$$

and j in the diagram (4.1) can be identified with the following composite of canonical maps:

$$(4.2) \quad \begin{aligned} Z_{\partial\Delta^{V-\alpha}} \times Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}} &\rightarrow Z_{\partial\Delta^{V-\alpha}} \ltimes (Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}}) \\ &\hookrightarrow (Z_{\partial\Delta^{V-\alpha}} \ltimes (Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}})) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}). \end{aligned}$$

Thus, the following homotopy equivalence holds: $Z_{(\Delta^V \cup_\alpha L)^\vee} \simeq (Z_{\partial\Delta^{V-\alpha}} \ltimes Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee})$. This homotopy equivalence and Example 2.2 induce the following homotopy equivalences:

$$\begin{aligned} Z_{(\Delta^V \cup_\alpha L)^\vee} &\simeq (Z_{\partial\Delta^{V-\alpha}} \ltimes Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}) \\ &\simeq Z_{\partial\Delta^{W-\alpha}} \vee (Z_{\partial\Delta^{V-\alpha}} \wedge Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}). \end{aligned}$$

By the above homotopy equivalence and the BBCG decomposition, it is easy to see that $Z_{(\Delta^V \cup_\alpha L)^\vee} \simeq \bigvee_{I \subset [m]} \Sigma|((\Delta^V \cup_\alpha L)^\vee)_I| \widehat{X}^I$. Here, we remark that we can apply Proposition 3.5 if $|W - \alpha| \geq 3$.

Similarly, if $K \neq \Delta^V$ and $L = \Delta^W$ then we have $Z_{(K \cup_\alpha \Delta^W)^\vee} \simeq (Z_{\partial\Delta^{V-\alpha}} \ltimes Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{K^\vee} \wedge Z_{\partial\Delta^{W-\alpha}}) \simeq \bigvee_{I \subset [m]} \Sigma|((K \cup_\alpha \Delta^W)^\vee)_I| \widehat{X}^I$.

Next, we consider the case that $K \neq \Delta^V$ and $L \neq \Delta^W$. Then, we have the push-out diagram

$$\begin{array}{ccc} (\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) & \longrightarrow & (\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee) \\ \downarrow & & \downarrow \\ (\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (K^\vee * \Delta^{W-\alpha}) & \longrightarrow & (K \cup_\alpha L)^\vee, \end{array}$$

which induces the following push-out diagram of spaces, by Lemmas 3.3 and 3.4:

$$\begin{array}{ccc} Z_{\partial\Delta^{V-\alpha}} \times Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}} & \longrightarrow & Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee)} \\ \downarrow & & \downarrow \\ Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (K^\vee * \Delta^{W-\alpha})} & \longrightarrow & Z_{(K \cup_\alpha L)^\vee}. \end{array}$$

Because $Z_{\partial\Delta^{V-\alpha}} \times Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}} \rightarrow Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee)}$ is a closed cofibration, $Z_{(K \cup_\alpha L)^\vee}$ is homotopy equivalent to the homotopy push-out of the following diagram:

$$\begin{array}{ccc} Z_{\partial\Delta^{V-\alpha}} \times Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}} & \xrightarrow{j_1} & Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee)} \\ \downarrow j_2 & & \\ Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (K^\vee * \Delta^{W-\alpha})} & & \end{array}$$

Because

$$\begin{aligned} Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (\Delta^{V-\alpha} * L^\vee)} &\simeq (Z_{\partial\Delta^{V-\alpha}} \ltimes (Z_{\Delta^\alpha} \times Z_{\partial\Delta^{W-\alpha}})) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}) \\ &\simeq (Z_{\partial\Delta^{V-\alpha}} \ltimes Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}), \\ Z_{(\partial\Delta^{V-\alpha} * \Delta^\alpha * \partial\Delta^{W-\alpha}) \cup (K^\vee * \Delta^{W-\alpha})} &\simeq ((Z_{\partial\Delta^{V-\alpha}} \times Z_{\Delta^\alpha}) \ltimes Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{K^\vee} \wedge Z_{\partial\Delta^{W-\alpha}}) \\ &\simeq (Z_{\partial\Delta^{V-\alpha}} \ltimes Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{K^\vee} \wedge Z_{\partial\Delta^{W-\alpha}}), \end{aligned}$$

and j_1 and j_2 are as described in (4.2), it follows that

$$Z_{(K \cup_\alpha L)^\vee} \simeq Q \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}) \vee \Sigma(Z_{K^\vee} \wedge Z_{\partial\Delta^{W-\alpha}}),$$

where Q is a homotopy push-out of the following diagram:

$$\begin{array}{ccc} Z_{\partial\Delta^{V-\alpha}} \times Z_{\partial\Delta^{W-\alpha}} & \longrightarrow & Z_{\partial\Delta^{V-\alpha}} \rtimes Z_{\partial\Delta^{W-\alpha}} \\ \downarrow & & \\ Z_{\partial\Delta^{V-\alpha}} \rtimes Z_{\partial\Delta^{W-\alpha}} & & \end{array}.$$

In the above diagram, the vertical and horizontal maps are the canonical collapsing maps. Then, it is easy to see that $Q \simeq Z_{\partial\Delta^{V-\alpha}} \wedge Z_{\partial\Delta^{W-\alpha}}$, and we have obtained a homotopy equivalence

$$(4.3) \quad Z_{(K \cup_\alpha L)^\vee} \simeq (Z_{\partial\Delta^{V-\alpha}} \wedge Z_{\partial\Delta^{W-\alpha}}) \vee \Sigma(Z_{\partial\Delta^{V-\alpha}} \wedge Z_{L^\vee}) \vee \Sigma(Z_{K^\vee} \wedge Z_{\partial\Delta^{W-\alpha}}).$$

To complete the proof, we now apply Proposition 3.5. Clearly, (4.3) implies that $Z_{(K \cup_\alpha L)^\vee}$ is a suspension space. Because α is not a facet of K or L , we have that $|V - \alpha| \geq 2$ and $|W - \alpha| \geq 2$. Then, by Example 2.2 we have that $Z_{\partial\Delta^{V-\alpha}} = Z_{\partial\Delta^{V-\alpha}}(C\underline{X}, \underline{X})$ and $Z_{\partial\Delta^{W-\alpha}} = Z_{\partial\Delta^{W-\alpha}}(C\underline{X}, \underline{X})$ are suspensions. Thus, (4.3) implies that $Z_{(K \cup_\alpha L)^\vee}$ is a double suspension, and so we complete the proof. \square

5. PRELIMINARIES FOR THE PROOF OF THEOREM 2.7

In this section, we review the Alexander duality and elementary collapses of simplicial complexes, and the Massey products of the Koszul homology.

5.1. Alexander duality. First, we review the Alexander duality of simplicial complexes. Although an elementary proof has been derived [4], we follow the classical argument presented in chapter six of Spanier's book [23].

Let M be a simplicial complex that is a subcomplex of $\partial\Delta^{n+1}$, and let L be a subcomplex of M . We denote the Alexander dual of L and M in $V(\partial\Delta^{n+1})$ by L^\vee , M^\vee . Then, M^\vee is a subcomplex of L^\vee , and the following duality is well-known, where we write $S^n = |\partial\Delta^{n+1}|$:

$$\begin{aligned} \gamma : H_q(|L^\vee|, |M^\vee|) &\cong H_q(|\text{Sd}L^\vee|, |\text{Sd}M^\vee|) \cong H_q(|\bar{L}|, |\bar{M}|) \\ &\cong H_q(S^n - |L|, S^n - |M|) \cong H^{n-q}(|M|, |L|) \end{aligned}$$

where $\text{Sd}K$ denotes the barycentric subdivision of K , \bar{L} is the supplement of L in $\partial\Delta^{n+1}$ defined in Definition 2.6.18 of [19], and the last isomorphism is the (topological) duality

$$\gamma_U : H_q(S^n - |L|, S^n - |M|) \cong H^{n-q}(|M|, |L|)$$

induced by an orientation class $U \in H^n(S^n \times S^n, S^n \times S^n - \Delta(S^n))$ of S^n , where $\Delta(S^n)$ is the diagonal set of $S^n \times S^n$.

Now, we consider the cohomology theory with coefficient $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and the relation between the duality map $\gamma : H_q(|L^\vee|, |M^\vee|; \mathbb{Z}_2) \rightarrow H^{n-q}(|M|, |L|; \mathbb{Z}_2)$ and the Steenrod squaring

operations Sq^i . For a pair of finite complexes (X, A) , the Steenrod squaring operations Sq_*^i on a homology theory are defined as follows:

$$\langle \alpha, Sq_*^i(\beta) \rangle = \langle Sq^i(\alpha), \beta \rangle \quad \text{for } \alpha \in H^q(X, A; \mathbb{Z}_2) \text{ and } \beta \in H_{q+i}(X, A; \mathbb{Z}_2),$$

where $\langle -, - \rangle$ denotes the Kronecker product.

Lemma 5.1.1. For $\alpha \in H_q(|L^\vee|, |M^\vee|)$ and $i > 0$, we have that

$$\gamma(Sq_*^i(\alpha)) = \sum_{j=0}^{i-1} Sq^{i-j}(\gamma(Sq_*^j(\alpha))),$$

Proof. In this proof, we omit the coefficient ring \mathbb{Z}_2 in the (co)homology theory. First, we remark that $Sq^i(U) = 0$ for $i > 0$. This follows from the fact that the natural restriction map $H^n(S^n \times S^n, S^n \times S^n - \Delta(S^n)) \rightarrow \tilde{H}^n(S^n \times S^n)$ is monomorphic. In the definition of the duality map γ , all maps except for γ_U commute with the squaring operations. Therefore, it suffices to prove the corresponding formula for γ_U .

Because $\gamma_U(Sq_*^i(\alpha)) \in H^{n-q+i}(|M|, |L|)$, we take an element $\beta \in H_{n-q+i}(|M|, |L|)$ and compute the Kronecker product $\langle \gamma_U(Sq_*^i(\alpha)), \beta \rangle$. Let

$$j : (|M|, |L|) \times (S^n - |L|, S^n - |M|) \rightarrow (S^n \times S^n, S^n \times S^n - \Delta(S^n))$$

be the inclusion map. Then,

$$\begin{aligned} \langle \gamma_U(Sq_*^i(\alpha)), \beta \rangle &= \langle j^*(U)/Sq_*^i(\alpha), \beta \rangle \\ &= \langle j^*(U), \beta \times Sq_*^i(\alpha) \rangle \\ &= \langle j^*(U), Sq_*^i(\beta \times \alpha) \rangle - \sum_{j=0}^{i-1} Sq_*^{i-j} \beta \times Sq_*^j(\alpha) \rangle \\ &= \langle Sq^i j^*(U), \beta \times \alpha \rangle + \sum_{j=0}^{i-1} \langle j^*(U), Sq_*^{i-j} \beta \times Sq_*^j(\alpha) \rangle \\ &= \sum_{j=0}^{i-1} \langle j^*(U)/Sq_*^j(\alpha), Sq_*^{i-j} \beta \rangle \\ &= \langle \sum_{j=0}^{i-1} Sq^{i-j}(\gamma_U(Sq_*^j(\alpha))), \beta \rangle, \end{aligned}$$

where in the fifth equation we use the fact that $Sq^i j^*(U) = j^* Sq^i(U) = 0$. Thus, the proof is completed. \square

5.2. Elementary collapse. In this subsection, we briefly review the notion of elementary collapse, which is necessary to prove Theorem 2.7.

Definition 5.2.1. A non-empty face σ in a simplicial complex K is a *free face* if it is not a facet of K and is contained in exactly one facet of K .

An *elementary collapse* of K is a simplicial complex K' obtained from K by the removal of a free face σ , along with all faces that contain σ . If there is a sequence of elementary collapses leading from K to K' , we say that K is collapsible onto K' , and we use the notation $K \searrow K'$.

Example 5.2.2. Let Δ^m denote the full simplex on the vertex set $[m+1]$. There exists a sequence of elementary collapses of $\Delta^m \boxtimes \Delta^1$ given by removing pairs of faces from the top to bottom in the following list:

$$\begin{aligned} & \{\underline{1} \cdots \underline{m+1}, \underline{1} \cdots \underline{m+1} \overline{m+1}\}, \\ & \dots \\ & \{\underline{1} \cdots \underline{i} \overline{i+1} \cdots \overline{m+1}, \underline{1} \cdots \underline{i} \overline{i+1} \cdots \overline{m+1}\}, \\ & \dots \\ & \{\underline{1} \overline{2} \cdots \overline{m+1}, \underline{1} \overline{2} \cdots \overline{m+1}\}, \end{aligned}$$

where $\underline{i} = (i, 1)$, $\overline{i} = (i, 2)$, and $\underline{1} \cdots \underline{i} \overline{i+1} \cdots \overline{m+1}$ denotes, for example, the face $\{\underline{1}, \dots, \underline{i}, \overline{i+1}, \dots, \overline{m+1}\}$ of $\Delta^m \boxtimes \Delta^1$. Thus, we see that $\Delta^m \boxtimes \Delta^1 \searrow (\partial \Delta^m \boxtimes \Delta^1) \cup (\Delta^m \boxtimes 2)$. By repeating the same process for smaller simplicies, we see that $\Delta^m \boxtimes \Delta^1 \searrow \Delta^m \boxtimes 2$. This can be applied to any simplicial complex K , and we see that $K \boxtimes \Delta^1$ is collapsible onto $K \boxtimes 2$.

5.3. Massey product in $\text{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$. Finally, we review the Massey products in $\text{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$, according to Section 3.2 of Buchstaber and Panov's book [5].

Recall that the torsion algebra $\text{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$ is defined as the homology group of the Koszul differential algebra $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[K], d)$, where their bigrading and the differentials are defined by $\deg u_i = (1, 2)$, $\deg v_i = (0, 2)$, and $du_i = v_i$, for $i = 1, \dots, m$. Here, we remark that we adopt the different conventions for the grading of $\text{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$.

A *multigrading* of the torsion algebra $\text{Tor}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$ is defined by setting

$$\text{mdeg } u_1^{\varepsilon_1} \cdots u_m^{\varepsilon_m} v_1^{i_1} \cdots v_m^{i_m} = (\varepsilon_1 + \cdots + \varepsilon_m, 2(i_1 + \varepsilon_1), \dots, 2(i_m + \varepsilon_m)),$$

and it is easy to see that

$$\text{Tor}_*^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) = \bigoplus_{\mathbf{a} \in \mathbb{N}^m} \text{Tor}_{*, 2\mathbf{a}}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}).$$

A subset $I \subset [m]$ may be viewed as a $(0, 1)$ -vector in \mathbb{N}^m , whose i -th coordinate is 1 if $i \in I$ and 0 otherwise. Then, the following multigraded version of Hochster's formula holds.

Theorem 5.3.1 (Theorem 3.2.9 of [5]). *For any subset $I \subset [m]$, we have that*

$$\text{Tor}_{i, 2I}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) \cong \tilde{H}^{|I|-i-1}(K_I; \mathbf{k}),$$

and $\text{Tor}_{i, 2\mathbf{a}}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) = 0$ unless \mathbf{a} is a $(0, 1)$ -vector.

Proposition 5.3.2. *If an n -fold Massey product $\langle a_1, \dots, a_n \rangle$ is defined for $a_i \in \text{Tor}_{*, 2I_i}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k})$, and $I_k \cap I_\ell \neq \emptyset$ for some $k \neq \ell$, then $\langle a_1, \dots, a_n \rangle$ is trivial.*

Proof. For $n = 2$, the assertion is well-known. Let $n > 2$, and consider an n -fold Massey product.

First, we recall the definition of the Massey product. Define \bar{a} to be $(-1)^{\deg a+1}a$. Then, the n -fold Massey product $\langle a_1, \dots, a_n \rangle$ is defined to be the set of all homology classes represented by elements of the form

$$a_{1,n} = \bar{a}_{1,1}a_{2,n} + \bar{a}_{1,2}a_{3,n} + \dots + \bar{a}_{1,n-1}a_{n,n},$$

for all solutions of the equations

$$\begin{aligned} a_i &= [a_{i,i}] \quad \text{for } 1 \leq i \leq n, \\ da_{i,j} &= \bar{a}_{i,i+1}a_{i+1,j} + \bar{a}_{i,i+2}a_{i+2,j} + \dots + \bar{a}_{i,j-1}a_{j,j} \quad \text{for } 1 \leq i \leq j \leq n, (i,j) \neq (1,n). \end{aligned}$$

Then, it is easy to see that $\langle a_1, \dots, a_n \rangle \cap \text{Tor}_{*, 2(I_1 + \dots + I_n)}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) \neq \emptyset$. If $I_k \cap I_\ell \neq \emptyset$ for some $k \neq \ell$, then $I_1 + \dots + I_n$ is not a $(0, 1)$ -vector. Therefore, it follows from Theorem 5.3.1 that $\text{Tor}_{*, 2(I_1 + \dots + I_n)}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K], \mathbf{k}) = 0$. Thus, we complete the proof. \square

Corollary 5.3.3. *Let $K = K_1 \cup_\alpha K_2$ be a simplicial complex on $[m]$, which is obtained from two simplicial complexes K_1 and K_2 by gluing along a common simplex α that is not equal to $V(K_1)$ or $V(K_2)$. Then, K^\vee is non-Golod over a field \mathbf{k} if and only if there are faces of σ and τ of K satisfying the following conditions:*

- (1) $V(K_1) - \alpha \subset \sigma$, $V(K_2) - \alpha \subset \tau$, and $\sigma \cup \tau = [m]$,
- (2) the inclusion map $\text{star}_{\text{link}_K(\sigma \cap \tau)}(\sigma - \sigma \cap \tau) \cup \text{star}_{\text{link}_K(\sigma \cap \tau)}(\tau - \sigma \cap \tau) \rightarrow \text{link}_K(\sigma \cap \tau)$ induces a non-trivial map in the homology theory with coefficient \mathbf{k} .

Proof. First, we show that higher Massey products in $\text{Tor}_*^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K^\vee], \mathbf{k})$ are trivial for any simplicial complex $K = K_1 \cup_\alpha K_2$.

Let $a_i \in \text{Tor}_{*, 2I_i}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K^\vee], \mathbf{k})$ for $i = 1, \dots, n$, where $n \geq 3$. Then, we want to prove that an n -fold Massey product $\langle a_1, \dots, a_n \rangle$ is trivial. If J is a face of K^\vee , then $\text{Tor}_{*, 2J}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K^\vee], \mathbf{k}) \cong \tilde{H}^*((K^\vee)_J; \mathbf{k}) = 0$. By this fact and the proposition above, we may assume that $I_i = \sigma_i^c$ for some simplex $\sigma_i \in K$ for $i = 1, \dots, n$, and $I_i \cap I_j = \emptyset$ for $i \neq j$. This implies that $\sigma_i \cup \sigma_j = [m]$. Thus, we may also assume that $v_0 \in I_1$, where $v_0 \in V(K_1) - \alpha$. Then, $v_0 \notin I_j = \sigma_j^c$, i.e., $v_0 \in \sigma_j$ for $j > 1$. Because a face of K containing the vertex v_0 is a subset of $V(K_1)$, we see that $V(K_2) - \alpha = V(K_1)^c \subset \sigma_j^c = I_j$ for $j > 1$, which contradicts the assumption that $I_j \cap I_k = \emptyset$ for $j \neq k > 1$. Thus, we have proved that higher Massey products in $\text{Tor}_{*, [m]}^{\mathbf{k}[v_1, \dots, v_m]}(\mathbf{k}[K^\vee], \mathbf{k})$ are trivial.

It follows that K^\vee is non-Golod if and only if there are disjoint subsets I and J of $[m]$ such that

$$\iota_{I,J} : (K^\vee)_{I \sqcup J} = (\text{link}_K(I^c \cap J^c))_{I \sqcup J}^\vee \rightarrow (K^\vee)_I * (K^\vee)_J = (\text{link}_K(I^c))_I^\vee * (\text{link}_K(J^c))_J^\vee$$

induces a non-trivial map in the cohomology theory. Then, I^c and J^c must be faces of K , and $I^c \cup J^c = [m]$, because $I \cap J = \emptyset$. Thus, we may assume that $V(K_1) - \alpha \subset I^c = \sigma$ and $V(K_2) - \alpha \subset J^c = \tau$. By the Alexander duality and its naturality, it follows that $\iota_{I,J}$ induces a non-trivial map in the cohomology theory if and only if the dual map

$$\begin{aligned} \iota_{I,J}^\vee : ((\text{link}_K(\sigma))^\vee * (\text{link}_K(\tau))^\vee)_{I \sqcup J}^\vee &= \text{link}_K(\sigma) * \Delta^J \cup \Delta^I * \text{link}_K(\tau) \\ &= \text{star}_{\text{link}_K(\sigma \cap \tau)}(\sigma - \sigma \cap \tau) \cup \text{star}_{\text{link}_K(\sigma \cap \tau)}(\tau - \sigma \cap \tau) \\ &\rightarrow \text{link}_K(\sigma \cap \tau) \end{aligned}$$

induces a non-trivial map in the homology theory, where the first equality follows from Lemma 3.1. Thus, we complete the proof. \square

Example 5.3.4. Let L be the simplicial complex on $\{2, 3, 4, 5, 6, 7\}$ with facets

$$267, 367, 467, 567, 236, 456, \alpha = 2345.$$

Then, the Alexander dual of $K = \Delta^{\alpha+1} \cup_\alpha L$ is not Golod. In fact, for $\sigma = 12345$ and $\tau = 67$, the inclusion map

$$\text{star}_K(\sigma) \cup \text{star}_K \tau = \Delta^{\alpha+1} \cup \{\emptyset, 2, 3, 4, 5\} * \Delta^{\{6,7\}} \rightarrow K$$

induces a non-trivial map in one dimensional homology groups. Needless to say, α is a facet of L .

6. PROOF OF THEOREM 2.7

In this section, we prove Theorem 2.7. The proof is divided into three parts. In the first part, we show that K^\vee is Golod. Next, we show that K^\vee is not stably homotopy Golod. Finally, we demonstrate that Z_{K^\vee} is torsion free for a particular K .

Before beginning the proof of Theorem 2.7, we study the topology of the space $|(S_k^3 \boxtimes \Delta^1) \cup_{\eta_k} S_4^2|$. As observed in Example 5.2.2, there exists a sequence of elementary collapses that collapses $S_k^3 \boxtimes \Delta^1$ onto S_k^3 , which induces a sequence of elementary collapses that collapses $(S_k^3 \boxtimes \Delta^1) \cup_{\eta_k} S_4^2$ onto S_4^2 . In particular, there exists a deformation retraction $\pi : |(S_k^3 \boxtimes \Delta^1) \cup_{\eta_k} S_4^2| \rightarrow |S_4^2|$ such that $\pi \circ |i| \simeq |\eta_k| : |S_k^3| \xrightarrow{|i|} |(S_k^3 \boxtimes \Delta^1) \cup_{\eta_k} S_4^2| \xrightarrow{\pi} |S_4^2|$, where i is the inclusion map.

6.1. K^\vee is Golod. Recall that

$$K = \Delta^{\alpha+v_0} \cup ((S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2) \cup F(S_k^3)) * \Delta^{\{w_1\}} \cup S_k^3 * \Delta^{\{w_1, w_2\}}.$$

Thus, by Corollary 5.3.3 we only have to prove that the map

$$\text{star}_{\text{link}_K(\sigma \cap \tau)}(\sigma - \sigma \cap \tau) \cup \text{star}_{\text{link}_K(\sigma \cap \tau)}(\tau - \sigma \cap \tau) \rightarrow \text{link}_K(\sigma \cap \tau)$$

induces the trivial map in the homology theory, where σ and τ are faces of K such that $\sigma \subset \alpha + v_0$, $\{w_1, w_2\} \subset \tau$, and $\sigma \cup \tau = V(K)$.

First, we consider the case with $\sigma \cap \tau = \emptyset$. That is, the map

$$\text{star}_K(\sigma) \cup \text{star}_K(\tau) = \Delta^{\alpha+v_0} \cup \text{star}_K(\tau) \rightarrow K$$

induces the trivial map in the homology theory. Because this map factors as $\Delta^{\alpha+v_0} \cup \text{star}_K(\tau) \rightarrow \Delta^{\alpha+v_0} \cup S_k^3 * \Delta^{\{w_1, w_2\}} \rightarrow K$, it suffices to prove that the inclusion map

$$\Delta^{v_0+\alpha} \cup S_k^3 * \Delta^{\{w_1, w_2\}} \rightarrow K$$

induces the trivial map in the homology theory. Because $|\Delta^{v_0+\alpha} \cup S_k^3 * \Delta^{\{w_1, w_2\}}| \simeq \Sigma|S_k^3| = S^4$ and $|K| \simeq \Sigma|S_4^2| = S^3$, the above inclusion map induces the trivial map in the homology theory.

Next, we consider the case that $\rho = \sigma \cap \tau$ is a non-empty face of S_k^3 . Because

$$\text{link}_K(\rho) = \Delta^{\{v_0\} \cup \alpha - \rho} \cup (\text{link}_{S_k^3 \boxtimes \Delta^1 \cup \eta_k S_4^2}(\rho) \cup F(\text{link}_{S_k^3}(\rho)) * \Delta^{w_1} \cup \text{link}_{S_k^3}(\rho) * \Delta^{\{w_1, w_2\}}),$$

the situation is similar to the case with $\rho = \emptyset$. Here, we are only required to prove that the map

$$f : |\text{link}_{S_k^3}(\rho)| \rightarrow |\text{link}_{S_k^3 \boxtimes \Delta^1 \cup \eta_k S_4^2}(\rho)|$$

induced by the inclusion induces the trivial map in the homology theory. f factors through a contractible space $|\text{link}_{S_k^3 \boxtimes \Delta^1}(\rho)|$. In fact, because $S_k^3 \boxtimes \Delta^1$ is a triangulation of $S^3 \times I$ and ρ is a non-empty simplex in the boundary, it follows that $\text{link}_{S_k^3 \boxtimes \Delta^1}(\rho)$ is a triangulation of a hemisphere. Thus, f clearly induces the trivial map in the homology theory, and we complete the proof. \square

6.2. K^\vee is not stably homotopy Golod. Next, we show that the map

$$(6.1) \quad |\iota_{\{w_1, w_2\}, \alpha+v_0}| : |K^\vee| \rightarrow |(K^\vee)_{\{w_1, w_2\}} * (K^\vee)_{\alpha+v_0}|$$

is stably non-trivial. To show this, we consider the mapping cone of $|\iota_{\{w_1, w_2\}, \alpha+v_0}|$, which is denoted by $C_{|\iota_{\{w_1, w_2\}, \alpha+v_0}|}$, and show that Sq^2 acts non-trivially on its mod-2 cohomology groups. In fact, we have the following isomorphisms of (co)homology groups:

$$\begin{aligned} \tilde{H}^p(C_{|\iota_{\{w_1, w_2\}, \alpha+v_0}|}; \mathbb{Z}_2) &\cong H^p(|(K^\vee)_{\{w_1, w_2\}} * (K^\vee)_{\alpha+v_0}|; \mathbb{Z}_2) \\ &\cong H_{m-1-p}(K, \Delta^{\alpha+v_0} \cup S_k^3 * \Delta^{\{w_1, w_2\}}; \mathbb{Z}_2) \\ &\cong H_{m-1-p}(\Delta^{\alpha+v_0} \cup (S_k^3 \boxtimes \Delta^1 \cup_{\eta_k} S_4^2) * \Delta^{\{w_1\}}, \Delta^{\alpha+v_0} \cup S_k^3 * \Delta^{\{w_1\}}; \mathbb{Z}_2) \\ &\cong \tilde{H}_{m-1-p}(\Sigma \mathbb{C}P^2; \mathbb{Z}_2), \end{aligned}$$

and Sq_*^2 acts non-trivially on $\tilde{H}_5(\Sigma \mathbb{C}P^2; \mathbb{Z}_2)$. Thus, it follows from Lemma 5.1.1 that Sq^2 acts non-trivially on $\tilde{H}^{m-6}(C_{|\iota_{\{w_1, w_2\}, \alpha+v_0}|}; \mathbb{Z}_2)$, which implies that the map (6.1) is stably non-trivial. \square

6.3. Z_{K^\vee} is torsion free. Finally, we will demonstrate that Z_{K^\vee} is torsion free if K is constructed using the simplicial map $\eta_{12} : S_{12}^3 \rightarrow S_4^2$ described in [18]. We believe that Z_{K^\vee} is torsion free in general, but we are currently unable to prove this.

By Theorem 1.1 and the Alexander duality, we have the following isomorphisms:

$$H^p(Z_{K^\vee}; \mathbb{Z}) \cong \bigoplus_{I \subset [m]} \tilde{H}^{p-|I|-1}((K^\vee)_I : \mathbb{Z}) \cong \bigoplus_{I \subset [m]} \tilde{H}_{2|I|-p-2}(\text{link}_K(I^c) : \mathbb{Z}),$$

where we have used the fact that $(K^\vee)_I = (\text{link}_K(I^c))^\vee$. Thus, to show that Z_{K^\vee} is torsion free, it suffices to show that every $\text{link}_K(\sigma)$ is torsion free for all faces σ of K . The longest part of the computation is the following, and the remaining parts are omitted.

Lemma 6.3.1. *$\text{link}_{S_{12}^3 \boxtimes \Delta^1 \cup_{\eta_{12}} S_4^2}(\sigma)$ is torsion free for any simplex σ in $S_{12}^3 \boxtimes \Delta^1 \cup_{\eta_{12}} S_4^2$.*

Proof. Set $L = S_{12}^3 \boxtimes \Delta^1 \cup_{\eta_{12}} S_4^2$. If $\sigma = \emptyset$, then clearly $\text{link}_L(\sigma) = L$ is torsion free.

Let v be a vertex of S_{12}^3 . The Mayer-Vietoris sequence associated with the decomposition $L = (L - v) \cup \text{star}_L(v)$ reduces to the long exact sequence

$$\cdots \rightarrow \tilde{H}_i(\text{link}_L(v)) \rightarrow \tilde{H}_i(L - v) \rightarrow \tilde{H}_i(L) \rightarrow \cdots.$$

The sequence of elementary collapses of $S_{12}^3 \boxtimes \Delta^1$ onto $S_{12}^3 \boxtimes 2$ given in Example 5.2.2 induces a sequence of elementary collapses of $S_{12}^3 \boxtimes \Delta^1 \cup_{\eta_{12}} S_4^2$ onto S_4^2 . Moreover, this sequence of elementary collapses induces a collapse of $L - v$ onto S_4^2 . This means that the inclusion map $|L - v| \rightarrow |L|$ is homotopy equivalent, and that $\tilde{H}_*(\text{link}_L(v)) = 0$.

To proceed further with the computation, we need to know the concrete structure of L . S_{12}^3 has twelve vertices $\{a_i, b_i, c_i, d_i\}_{i=0,1,2}$, and the following is the list of its facets:

$$\begin{array}{cccccccc} a_0 b_0 c_0 c_1 & a_0 b_0 b_1 c_1 & a_0 a_1 b_1 c_1 & a_1 a_2 b_1 c_1 & a_2 b_1 c_1 c_2 & a_2 b_1 b_2 c_2 & a_2 b_0 b_2 c_2 & a_0 a_2 b_0 c_2 & a_0 b_0 c_0 c_2 \\ a_0 a_2 b_0 d_1 & a_0 b_0 b_1 d_1 & b_0 b_1 c_1 d_1 & b_1 c_1 c_2 d_1 & a_2 c_1 c_2 d_1 & a_0 a_2 c_2 d_1 & a_0 b_1 d_0 d_1 & b_1 c_2 d_0 d_1 & a_0 c_2 d_0 d_1 \\ a_0 b_1 d_0 d_2 & b_1 c_2 d_0 d_2 & a_0 c_2 d_0 d_2 & a_0 a_1 b_1 d_2 & a_1 a_2 b_1 d_2 & a_2 b_1 b_2 d_2 & a_2 b_0 b_2 d_2 & b_1 b_2 c_2 d_2 & b_0 b_2 c_2 d_2 \\ b_0 c_0 c_2 d_2 & b_0 c_0 c_1 d_2 & a_0 c_0 c_2 d_2 & a_0 c_0 c_1 d_2 & a_0 a_1 c_1 d_2 & a_1 a_2 c_1 d_2 & a_2 b_0 d_1 d_2 & b_0 c_1 d_1 d_2 & a_2 c_1 d_1 d_2 \end{array}$$

We order the vertices as $a_0 < a_1 < a_2 < b_0 < \cdots < c_0 < \cdots < d_0 < d_1 < d_2$. The vertices of S_4^2 are a, b, c, d , and are ordered such that $a < b < c < d$. Furthermore, $\eta_{12} : S_{12}^3 \rightarrow S_4^2$ is the map defined by the mappings $a_i \mapsto a$, $b_i \mapsto b$, $c_i \mapsto c$, and $d_i \mapsto d$, for $i = 0, 1, 2$. Then, the list

of the facets of L is as follows:

$a_0b_0c_0c_1c$			$a_0a_2b_0d_1d$	$a_0a_2b_0bd$	a_0a_2abd	$a_0b_1d_0d_2d$		
$a_0b_0b_1c_1c$	$a_0b_0b_1bc$		$a_0b_0b_1d_1d$	$a_0b_0b_1bd$		$b_1c_2d_0d_2d$		
$a_0a_1b_1c_1c$	$a_0a_1b_1bc$	a_0a_1abc	$b_0b_1c_1d_1d$	$b_0b_1c_1cd$	b_0b_1bcd	$a_0c_2d_0d_2d$		
$a_1a_2b_1c_1c$	$a_1a_2b_1bc$	a_1a_2abc	$b_1c_1c_2d_1d$	$b_1c_1c_2cd$		$a_0a_1b_1d_2d$	$a_0a_1b_1bd$	a_0a_1abd
$a_2b_1c_1c_2c$			$a_2c_1c_2d_1d$	$a_2c_1c_2cd$		$a_1a_2b_1d_2d$	$a_1a_2b_1bd$	a_1a_2abd
$a_2b_1b_2c_2c$	$a_2b_1b_2bc$		$a_0a_2c_2d_1d$	$a_0a_2c_2cd$	a_0a_2acd	$a_2b_1b_2d_2d$	$a_2b_1b_2bd$	
$a_2b_0b_2c_2c$	$a_2b_0b_2bc$		$a_0b_1d_0d_1d$			$a_2b_0b_2d_2d$	$a_2b_0b_2bd$	
$a_0a_2b_0c_2c$	$a_0a_2b_0bc$	a_0a_2abc	$b_1c_2d_0d_1d$			$b_1b_2c_2d_2d$	$b_1b_2c_2cd$	b_1b_2bcd
$a_0b_0c_0c_2c$			$a_0c_2d_0d_1d$			$b_0b_2c_2d_2d$	$b_0b_2c_2cd$	b_0b_2bcd
						$b_0c_0c_2d_2d$	$b_0c_0c_2cd$	
						$b_0c_0c_1d_2d$	$b_0c_0c_1cd$	
						$a_0c_0c_2d_2d$	$a_0c_0c_2cd$	
						$a_0c_0c_1d_2d$	$a_0c_0c_1cd$	
						$a_0a_1c_1d_2d$	$a_0a_1c_1cd$	a_0a_1acd
						$a_1a_2c_1d_2d$	$a_1a_2c_1cd$	a_1a_2acd
						$a_2b_0d_1d_2d$		
						$b_0c_1d_1d_2d$		
						$a_2c_1d_1d_2d$		

We see that $L = L_1 \cup L_2$, where L_1 is the subcomplex generated by the facets on the first to third columns from the left in the above list, and L_2 is the subcomplex generated by the other facets. Then, $L = \text{star}_L(c) \cup \text{star}_L(d)$. If $\sigma \cap \{c, d\} = \emptyset$, then $\text{link}_L(\sigma) = \text{link}_{L_1}(\sigma + c) * c \cup \text{link}_{L_2}(\sigma + d) * d$, which implies that $|\text{link}_L(\sigma)|$ is homotopy equivalent to a suspension space. Therefore, if $\text{link}_L(\sigma) \leq 2$, i.e., $\dim \sigma \geq 1$, then it follows that $\text{link}_L(\sigma)$ is torsion free, by dimensional reasoning. We consider a face with $\dim \sigma \leq 0$. For a vertex or the empty face in S_{12}^3 , we have already proved that $\text{link}_L(\sigma)$ is torsion free. If $v = a$ or $v = b$, then we see that $|\text{link}_L(v)|$ is homotopy equivalent to S^3 . Thus, we have completed the proof in this case.

If $c \in \sigma$ or $d \in \sigma$, then we see by direct computation that $|\text{link}_L(c)|$ and $|\text{link}_L(d)|$ are homotopy equivalent to S^3 . Moreover, $\text{link}_L(ac)$, $\text{link}_L(bc)$, $\text{link}_L(cd)$, $\text{link}_L(xc)$, $\text{link}_L(ad)$, $\text{link}_L(bd)$, and $\text{link}_L(xd)$ are triangulations of S^2 , where x is a vertex of S_{12}^3 . If $\dim \text{link}_L(\sigma) \leq 1$, then $\text{link}_L(\sigma)$ is torsion free by dimensional reasoning, and the proof is complete for all cases.

□

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DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES, OSAKA PREFECTURE UNIVERSITY, SAKAI,
599-8531, JAPAN

E-mail address: kiriye@mi.s.osakafu-u.ac.jp

DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES, OSAKA PREFECTURE UNIVERSITY, SAKAI,
599-8531, JAPAN

E-mail address: su301034@osakafu-u.net